PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Classical Field Theory of Matter and Electricity I. An Approach from First Principles

S. R. Milner

Phil. Trans. R. Soc. Lond. A 1960 **253**, 185-204
doi: 10.1098/rsta.1960.0021

THE CLASSICAL FIELD THEORY OF MATTER AND ELECTRICITY

I. AN APPROACH FROM FIRST PRINCIPLES

By S. R. MILNER, F.R.S.

(*First submitted* 31 *May* 1954—*in revised form* 25 *February* 1959)

The late S. R. Milner, F.R.S., Professor Emeritus of the University of Sheffield, who died at Sydney on 12 *August* 1958, *left his scientific papers to my care. Among these was the present revised and greatly simplified version of the derivation of the extended field equations of electromagnetic theory, which he wrote in* 1957. *He had no time to revise the second part, the Electromagnetic Theory of Particles. This is published in the original form, as first submitted in* 1954, *except for the necessary changes in references to the first part, and a few pencilled marginal notes which the author wrote in June* 1958, *shortly before his death.*

D. GABOR, F.R.S.

CONTENTS

The most desirable classical field theory of the fundamental continuous substratum of matter, from which we can imagine particles are formed, would generally be considered to be the electromagnetic equations but for the fact that these are not consistent with the permanent existence of electrons. Instead of attempting (as has been usual) to modify the equations by special assumptions for the purpose, the problem is attacked here by deriving from first principles field equations which represent *conserved* matter; for the failure of the standard equations can be traced to the fact that they do not admit conservation of energy and momentum in general, but only in simple cases.

The new equations are found to be identical with those of standard electromagnetic theory except that they contain two extra variables, which indicate the existence of additional energy, momentum and stress in the field. The two variables, however, come into the equations in a way which allows them to be included in the charge and current terms, so that they become there concealed and leave the form of the equations virtually unchanged. Consequently they do not affect the ordinary practical use which is made of the electromagnetic equations; they only come into open play in fundamental theory and in the presence of charge and current in the field, and there they remove the difficulties which the electromagnetic field theory in its accepted form presents.

1. Introduction

In standard relativity the theory of matter extended in space is usually confined to regarding it as a smoothed assemblage of particles and discussing its mechanical features, the energy and momentum densities and associated properties. To get more insight into the constitution of the particles themselves a 'field' theory of what we may think of as the fundamental continuous matter from which they are formed is also required. This would express the properties of the matter in terms of field variables and differential equations which show how their rates of variation are related. The classical electromagnetic theory seemed just what was wanted for this, but it has turned out to be defective for the purpose. It appears to be impossible on the Maxwell–Lorentz equations as they stand to devise an electron of finite mass that will hold its charge together, or even possess accurately the mechanical properties required of a particle by relativity theory.

The reason for this failure is, I think, that the well-known tensor which expresses the mechanical properties of the electromagnetic field in the classical theory gives only a part of the energy, momentum and stress which *can* be allocated to a field. Without some enlargement of the tensor it is impossible to make the energy and momentum in the field conserved when charge and current are present. In this part the problem is discussed of deriving in the form of a four-square matrix the mechanical tensor of an arbitrary complex vector field in 4-space. Such a field has more variables than the six electric and magnetic force components of the classical theory, so it evidently gives scope for the enlargement; but the determination of the actual terms of the tensor becomes now uncertain unless based on first principles. One that suffices is that matter is completely defined by an assemblage in matrix form of its essential properties; its proper mass must then be an invariant scalar magnitude characteristic of the matrix. The further consideration is much facilitated if the matrix is expressed in such co-ordinates that it represents a geometrical construct in Euclidean 4-space; its terms then denote the *components* of the construct, and the magnitude of their geometrical resultant can be identified with the proper density of the matter. It then appears that there are two different related matrix forms available, which have the same resultant magnitude, one which represents the variables of an arbitrary vector field, and the other of a kind which can denote the mechanical properties of the field. These matrices are necessarily consistent with each other, since each can be regarded as being formed by a decomposition into different kinds of components of the same portion of fundamental matter.

When the additional energy, momentum and stress which are obtained in this way are taken into account, the logical difficulties which prevent the standard electromagnetic equation from being employed as a field theory of fundamental matter are found to disappear.

2. The representation of physical quantities in a Euclidean fourfold

The development of the method outlined in the preceding section is carried out in § 3; this section contains an account of some mathematical results which are required in it.

In the restricted theory of relativity, to which the treatment here is confined, the location of a point in time and space can be made by a 'space-time' vector from the origin,

$$\mathrm{x} = |(\mathrm{x}_1 \,..\, \mathrm{x}_4) = |(ct, x, y, z), \tag{2·1a}$$

which is subject to a geometry defined by its magnitude x_0 being

$$\mathrm{x}_0 = (c^2t^2 - x^2 - y^2 - z^2) = (\bar{\mathrm{x}}\eta\mathrm{x})^{\frac{1}{2}}, \tag{2·1b}$$

where η is a diagonal matrix of the terms $(1, -1, -1, -1)$. By writing

$$x = |(x_1 \,..\, x_4) = |(ct, \mathrm{i}x, \mathrm{i}y, \mathrm{i}z) = \zeta\mathrm{x}, \tag{2·2a}$$

where ζ is the diagonal matrix of $(1, \mathrm{i}, \mathrm{i}, \mathrm{i})$, the location can be represented in a fourfold with Euclidean geometry

$$x_0 = (\bar{x}x)^{\frac{1}{2}} = \mathrm{x}_0. \tag{2·2b}$$

In the above a matrix notation for vectors is used. A plain smaller letter, x, or more generally a, stands for a *column* of the four components; when expressed in full it is written, to save space, along the line of print, and marked by a vertical stroke in front. The symbols denoting vectors and four-square matrices in Euclidean 4-space are written in italics, small and capital, respectively; in matrix formulae applying to space-time corresponding roman letters are used. The transpose of a 4^2 matrix (rows and columns interchanged) is denoted by a bar above the letter; correspondingly $\bar{x}$, $\bar{a}$ stand for rows of the components. According to ordinary matrix multiplication rules $\bar{a}b$ is the scalar product

$$\bar{a}b = a_i b_i = a_1 b_1 \,..\, a_4 b_4, \tag{2·3a}$$

but $a\bar{b}$ is the full matrix

$$ab = [a_i b_k] \quad (i, k = 1 \,..\, 4) \tag{2·3b}$$

(the square brackets are used to distinguish the matrix as a whole from its representative term).

While representation in space-time and Euclidean 4-space can be treated by the same formulae in tensor calculus, in the initial calculations here a material gain of simplicity is gained by using the Euclidean form only. The use of matrix notation in the formulae and the enumeration of the 1234 components as ct, ix, iy, iz instead of the more common x, y, z, ict is required for the same purpose. The final formulae, however, are converted into standard tensor forms.

The method of representation in a fourfold is commonly restricted to particle-events, but it may be applied with advantage to other physical quantites which can be denoted by vectors or systems of vectors. Among other things it ensures that their definitions are in full accord with the principle of relativity. The simple example which follows brings out some useful points. Let us represent at a point in the fourfold the mass (in energy units) of a particle at rest in our space by a vector p_0, whose magnitude is $p_0 = m_0c^2$, drawn parallel to the 'time' axis, 1. If p^0 be rotated about the point so as to become p, it will now denote the mass of the particle in another state. Since angles between x_1 and x_2, x_3, x_4 are imaginary, p must take the form

$$p = |(p_1 \,..\, p_4) = |(p_t, \mathrm{i}p_x, \mathrm{i}p_y, \mathrm{i}p_z) \tag{2·4}$$

when expressed explicitly in real physical terms; and, since the components must all have the same dimensions, it can be identified with $|(w, \mathrm{i}cg_x, \mathrm{i}cg_y, \mathrm{i}cg_z)$, where w and $\mathfrak{g}$ are the energy and the momentum 3-vector. The magnitude of p being unchanged by the rotation, we have

$$p_0 = (\bar{p}p)^{\frac{1}{2}} = (w^2 - c^2g^2) = m_0c^2.$$

23-2

The points brought out by this example are: first, that the mass of a particle is no more than the resultant of the physical components into which it has been decomposed by representation in the fourfold; and secondly, that vectors representing physical quantities generally, as well as displacements, will possess a simplified kind of complexity in that their components are not all real. The first point gives some justification for taking over the same idea here, where *the proper density of fundamental matter is regarded as the resultant magnitude of the* 16 *terms of a matrix into which it can be decomposed* in a similar way to that of the example. Not only is the second point well recognized, but it is, I think, commonly accepted without question that vectors like (2·4), the components of which are initially defined as simple, maintain this simple form however they may be rotated in the fourfold. It has now to be observed that this view requires some amendment, and that *an originally simple vector can be imagined rotated to such an orientation that its components must be denoted by fully complex numbers.* As this fact is of special consequence in the developments of § 3, a brief account of the relevant rotation theory is here given. While the general principles are well known, I have not been able to find in the literature some of the results required.

In what follows R, S stand for the columns of matrices,

$$\mathrm{R} = \begin{pmatrix} 1 & . & . & . & . & -1 & . & . & . & . & -1 & . & . & . & . & -1 \\ . & 1 & . & . & 1 & . & . & . & . & . & . & 1 & . & . & -1 & . \\ . & . & 1 & . & . & . & . & -1 & 1 & . & . & . & . & 1 & . & . \\ . & . & . & 1 & . & . & 1 & . & . & -1 & . & . & 1 & . & . & . \end{pmatrix}, \tag{2·5a}$$

$$\mathrm{S} = \begin{pmatrix} 1 & . & . & . & . & 1 & . & . & . & . & 1 & . & . & . & . & 1 \\ . & 1 & . & . & -1 & . & . & . & . & . & . & 1 & . & . & -1 & . \\ . & . & 1 & . & . & . & . & -1 & -1 & . & . & . & . & 1 & . & . \\ . & . & . & 1 & . & . & 1 & . & . & -1 & . & . & -1 & . & . & . \end{pmatrix}, \tag{2·5b}$$

and $\mathrm{R}_{\alpha ij}$ denotes the term in the ith row and the jth column of the αth matrix of the column set R. R and S form the simplest sets of matrices having the multiplication properties of the 1, i, j, k of quaternions; the matrices R_α, S_β possess also the following further properties which are used later:

(*a*) Of orthogonality—the reciprocal is the same as the transpose:

$$(\mathrm{R}_\alpha)^{-1} = \overline{\mathrm{R}_\alpha}; \quad [\mathrm{R}^{-1}_{\alpha ij} = R_{\alpha ji}]. \tag{2·6a}$$

(It is convenient to write R^{-1} as a symbol for a column of the reciprocals $|(\mathrm{R}_1^{-1} \ldots \mathrm{R}_4^{-1})$, and similarly for S.) By the properties of quaternions we have also

$$\mathrm{R}_\alpha^{-1} = (\eta \mathrm{R})_\alpha, \quad [(\eta \mathrm{R})_{\alpha ik} = (\eta_{\alpha j} \mathrm{R}_j)_{ik}].$$

(*b*) Of being convertible into each other by means of η

$$\mathrm{S}_\alpha = \eta \mathrm{R}_\alpha \eta, \quad \mathrm{R}_\alpha = \eta \mathrm{S}_\alpha \eta, \quad [(\eta \mathrm{R}_\alpha)_{ik} = \eta_{ij} \mathrm{R}_{\alpha jk}]. \tag{2·6b}$$

(*c*) Of being convertible by transposing suffixes thus

$$\mathrm{R}_{\alpha ij} = \mathrm{S}_{j\alpha i} = (\mathrm{R}_i \eta)_{\alpha j}; \quad (\mathrm{R}_\alpha \mathrm{S}_\beta)_{ik} = (\mathrm{R}_i \mathrm{S}_k)_{\alpha\beta}. \tag{2·6c}$$

(*d*) Of commuting in mutual products

$$\mathrm{R}_\alpha \mathrm{S}_\beta = \mathrm{S}_\beta \mathrm{R}_\alpha. \tag{2·6d}$$

In the same way as in (2·3) the scalar product of a vector a and the column R, $a_\alpha R_\alpha$, may be written shortly as $\bar{a}$R. This is, of course, a 4^2 matrix, and its value and that of a similarly formed $\bar{b}$S $(= b_\beta S_\beta)$ are given in full below

$$\bar{a}R = \begin{bmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & -a_4 & a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{bmatrix}, \quad \bar{b}S = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ -b_2 & b_1 & -b_4 & b_3 \\ -b_3 & b_4 & b_1 & -b_2 \\ -b_4 & -b_3 & b_2 & b_1 \end{bmatrix}. \tag{2·7}$$

The matrix $\bar{a}$R, read by either rows or columns, denotes a set of four perpendicular lines, each of the same magnitude $a_0 = (\bar{a}a)^{\frac{1}{2}}$; it will be called an orthogonal *vector system*. The resultant magnitude of the whole system, defined as the square root of the sum of the squares of all the terms of the matrix, is $2a_0$. $\bar{b}$S is a similar but differently constructed system, of resultant magnitude $2b_0$.

(*a*) *Rotation in the fourfold*

(i) *General and constituent rotations*

If c and d are independent unit vectors, $\bar{c}$R and $\bar{d}$S are each orthogonal, and $\bar{c}$R$\bar{d}$S $(=\bar{d}S\,\bar{c}R)$, having six independent variables, is a form of the general orthogonal matrix Ω in four dimensions. When Ω is pre-multiplied into a matrix denoting an orthogonal vector system it rotates the column vectors to arbitrary orientations, consistent with their mutual perpendicularity, and when post-multiplied the row vectors.

The first constituent of Ω, $\bar{c}$R, with three independent variables, pre-multiplied into a given column vector drawn from the origin, will rotate it to an arbitrary direction. The nature of the rotation can readily be seen by putting c_3, c_4, zero, and $c_1, c_2, = \cos\theta, \sin\theta$, when it is found to consist of simultaneous rotations through the same angle θ in the planes 12 and 34. Similar equal rotations in dual axial planes 13, 42; 14, 23 result when c_2, c_4; c_2, c_3 are put zero, and when all the components of c are finite, the resultant effect becomes a rotation in some plane containing the axis 1, accompanied by an equal rotation in the plane dual to this.

If we write

$$c(\theta, \phi, \psi) = |(\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi\cos\psi, \sin\theta\sin\phi, \sin\psi),$$

it is easy to verify that $\quad \overline{c(\theta', \phi, \psi)}\,R\,\overline{c(\theta, \phi, \psi)}\,R = \overline{c(\theta+\theta', \phi, \psi)}\,R, \tag{2·8}$

from which it follows that ϕ and ψ determine the plane, while θ expresses the angle, of rotation. In order to convert time-like components of a rotated vector into space-like ones, θ must be imaginary; c then takes on the usual form for a vector

$$c = |(c_1 \,..\, c_4) = |(c_t \mathrm{i}c_x \mathrm{i}c_y \mathrm{i}c_z);$$

also $$c_0^2 = c_t^2 - c_x^2 - c_y^2 - c_z^2 = 1. \tag{2·9}$$

(ii) *Space rotations*

The other constituent of Ω, $\bar{d}$S, denotes similar 'locked' rotations; in fact, $\bar{c}$S is the same as $\bar{c}$R, except that the rotation in the plane containing the axis 1, is in the opposite direction. Consequently, if we degenerate Ω by making $d = c$, so that it takes the form $\bar{c}$R$\,\bar{c}$S, the two rotations in the plane containing axis 1 cut each other out, while those in the dual plane combine in accordance with (2·8) to give a resultant rotation of 2θ. Since the dual plane lies wholly in the 3-space 234, $\bar{c}\mathrm{R}\,\bar{c}\mathrm{S}$ denotes a rotation confined to this space. This is shown by its value in full, with c as in (2·9), given below

$$\bar{c}\mathrm{R}\,\bar{c}\mathrm{S}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & c_t^2-c_x^2+c_y^2+c_z^2 & 2(-c_xc_y-\mathrm{i}c_tc_z) & 2(-c_xc_z+\mathrm{i}c_tc_y)\\ 0 & 2(-c_xc_y+\mathrm{i}c_tc_z) & c_t^2+c_x^2-c_y^2+c_z^2 & 2(-c_yc_z-\mathrm{i}c_tc_x)\\ 0 & 2(-c_xc_z-\mathrm{i}c_tc_y) & 2(-c_yc_z+\mathrm{i}c_tc_x) & c_t^2+c_x^2+c_y^2-c_z^2\end{bmatrix}. \qquad (2\cdot10)$$

When multiplied into a vector $a=|(a_t\mathrm{i}a_x\mathrm{i}a_y\mathrm{i}a_z)$ the matrix (2·10) affects only the space components; being, however, complex it changes them from simple into complex terms. This is due to θ being imaginary; if θ is taken to be real the 3^2 matrix which (2·10) virtually forms (now real throughout) has been known for a long time (cf. Turnbull 1929) as the expression of an arbitrary rotation in 3-space.

(iii) *Lorentz rotations*

In $\bar{c}\mathrm{S}^{-1}$, which is the same as the transpose of $\bar{c}\mathrm{S}$, and so is readily obtained from (2·7), the directions of the locked rotations are reversed from those of $\bar{c}\mathrm{S}$. Consequently the rotation produced by $\bar{c}\mathrm{R}\,\bar{c}\mathrm{S}^{-1}$ is 2θ in a plane containing the axis 1 and a line from the origin in the 234-space defined by ϕ and ψ, with no rotation in the 3-space. This rotation, with c as in (2·9) applied to the vector x (2·2a), is written out in full below

$$x'=\bar{c}\mathrm{R}\,\bar{c}\mathrm{S}^{-1}x, \qquad (2\cdot11)$$

$$\begin{bmatrix}ct'\\ \mathrm{i}x'\\ \mathrm{i}y'\\ \mathrm{i}z'\end{bmatrix}=\begin{bmatrix}c_t^2+c_x^2+c_y^2+c_z^2 & -2\mathrm{i}c_tc_x & -2\mathrm{i}c_tc_y & -2\mathrm{i}c_tc_z\\ 2\mathrm{i}c_tc_x & c_t^2+c_x^2-c_y^2-c_z^2 & 2c_xc_y & 2c_xc_z\\ 2\mathrm{i}c_tc_y & 2c_xc_y & c_t^2-c_x^2+c_y^2-c_z^2 & 2c_yc_z\\ 2\mathrm{i}c_tc_z & 2c_xc_z & 2c_yc_z & c_t^2-c_x^2-c_y^2+c_z^2\end{bmatrix}\begin{bmatrix}ct\\ \mathrm{i}x\\ \mathrm{i}y\\ \mathrm{i}z\end{bmatrix}.$$

The i's disappear on multiplying out, and real equations for ct', x', y', z' in terms of $c_t\,..\,c_z$, $ct\,..\,z$ follow. Alternatively (2·11) can be written in space-time terms direct

$$\mathrm{x}'=\zeta^{-1}x'=\zeta^{-1}\bar{c}\mathrm{R}\,\bar{c}\mathrm{S}^{-1}\zeta\mathrm{x}. \qquad (2\cdot12)$$

This form removes all the i's, and the minus signs from the first row, without other alteration.

On substituting, we have

$$c_t=\frac{1}{\surd 2}\left(\frac{1}{\surd(1-v^2/c^2)}+1\right)^{\frac{1}{2}},\quad c_{x,y,z}=\frac{1}{\surd 2}\frac{v_{x,y,z}}{v}\left(\frac{1}{\surd(1-v^2/c^2)}-1\right)^{\frac{1}{2}}, \qquad (2\cdot13)$$

where $v=(v_x^2+v_y^2+v_z^2)$, (2·11) and (2·12) become identifiable as a Lorentz transformation of co-ordinates in which the axes of the unaccented system are in motion with a velocity $\mathbf{v}$ relative to the accented system, the systems coinciding at $t=t'=0$ (cf. Pauli 1921, where the transformation is given in 3-vector notation).

As ordinarily used the Lorentz transformation (2·12) operates on the space and time co-ordinates of the system, and so it must always be applied to the system as a whole. In the representation of this process in the fourfold, the system is regarded as fixed, but it is now referred to the accented axes which have been rotated backward with respect to the unaccented, i.e. by $(\bar{c}\mathrm{R}\,\bar{c}\mathrm{S}^{-1})^{-1}$. Rotations are, however, used in a different way here. The objective is to construct in the fourfold vector fields (fields marked at every point by a scalar magnitude and a direction of its 'action') which may be expected to describe various aspects of fundamental matter. The direction at any point can be specified by imagining a small element of the field there to have been rotated from some standard direction (one of the axes) to its direction in the actual field. Such imagined local rotations clearly have no effect on the co-ordinate system, and so are not tensor transformations.

The special feature of the foregoing formulation of rotation matrices in 4-space is that it shows them as products of *two independent commuting factors* $\bar{c}\mathrm{R}$ *and* $\bar{d}\mathrm{S}$, *so that any rotation is decomposed into two elementary rotations, which can be carried out in either order*. The elementary rotations are of the 'locked' type—simultaneous equal rotations in dual planes—first brought to light by Eddington in his analysis of E-numbers. They seemed there, one must admit, rather abstruse conceptions, being treated in a symbolic 16-space, so it is interesting to find them presenting themselves as fundamental constituents of the simple real rotations in 4-space employed in ordinary relativity theory.

(*b*) *Complex vectors*

The fact just mentioned is of some importance here, for these locked rotations, when applied to a vector whose components have the usual simple, but not all real, form of $x = |(ct\,\mathrm{i}x\,\mathrm{i}y\,\mathrm{i}z)$, say $q = |(q, \mathrm{i}q_x, \mathrm{i}q_y, \mathrm{i}q_z)$, convert the components into fully complex quantities. To see this consider $q' = \bar{c}\mathrm{R}q$ in the simplest case, in which $c = |(\cos\theta, \sin\theta, 0, 0)$; it gives

$$\left.\begin{aligned} q'_t &= \cos\theta\, q_t - \mathrm{i}\sin\theta\, q_x, & q'_x &= \cos\theta\, q_x - \mathrm{i}\sin\theta\, q_t,\\ q'_y &= \cos\theta\, q_y - \sin\theta\, q_z, & q'_z &= \cos\theta\, q_z + \sin\theta\, q_y. \end{aligned}\right\} \tag{2·14}$$

Hence if θ is real, q'_t and q'_x are complex; if imaginary, q'_y and q'_z. In combined R- and S-rotations also, except for the two cases of rotation in a single plane, $\bar{c}\mathrm{R}\,\bar{c}\mathrm{S}$ with θ real, the rotation in 3-space, and $\bar{c}\mathrm{R}\,\bar{c}\mathrm{S}^{-1}$ with θ imaginary, the Lorentz rotation, the terms of q will be changed from simple to complex. Moreover, the general rotation, $\Omega = \bar{c}\mathrm{R}\,\bar{d}\mathrm{S}$, so long as c and d are independent and not both real, is itself a fully complex matrix.

Nevertheless, Ω remains orthogonal, and this gives a special form to the expression for the magnitude, supposed real, of a complex vector field in the fourfold. Ωq always implies a conjugate vector $\Omega^* q^*$, in which every i, explicitly stated or not, is changed in sign. But, while $\bar{q}q^*$ is the simplest real scalar formed from q and q^*, it is not invariant; for in

$$\overline{q'}q^{*\prime} = \overline{\Omega q}\,\Omega^* q^* = \bar{q}\bar{\Omega}\,\Omega^* q^*$$

Ω^* and $\bar{\Omega}$ do not cancel as would Ω and $\bar{\Omega}$. Further, $\bar{q}q$ and $\overline{q^*}q^*$, though invariant, are still complex scalars; if we put

$$q = \rho + \mathrm{i}\sigma, \quad q^* = \rho - \mathrm{i}\sigma, \tag{2·15}$$

where ρ, σ are columns of four real component quantities, then

$$\bar{q}q = (\bar{\rho}\rho - \bar{\sigma}\sigma) + 2\mathrm{i}\bar{\rho}\sigma, \quad \overline{q^*}q^* = (\bar{\rho}\rho - \bar{\sigma}\sigma) - 2\mathrm{i}\bar{\rho}\sigma.$$

Consequently $(\bar{\rho}\rho - \bar{\sigma}\sigma)$ and $\bar{\rho}\sigma$ separately are invariant, but both together are required to get a real invariant fully characteristic of a vector field of q. This is given by

$$(\bar{q}q\,\overline{q^*}q^*)^{\frac{1}{2}} = \{(\bar{\rho}\rho - \bar{\sigma}\sigma)^2 + 4(\bar{\rho}\sigma)^2\}^{\frac{1}{2}} \tag{2·16}$$

on taking it to have the dimensions of $\bar{q}q^*$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

A complex vector can also be regarded as the sum of two antithetic vectors of the standard simple term form. Let

$$q = r + \mathrm{i}s, \quad q^* = r^* - \mathrm{i}s^*, \tag{2·17}$$

where

$$r = |(r_t \mathrm{i}r_x \mathrm{i}r_y \mathrm{i}r_z), \quad s = |(s_t \mathrm{i}s_x \mathrm{i}s_y \mathrm{i}s_z) \tag{2·17a}$$

are simple-term vectors like x, so that in a description in space and time $r_t \ldots r_z$, $s_t \ldots s_z$ denote real components. The scheme expressing q in terms of these is

$$\left.\begin{aligned} q &= |(r_1 + \mathrm{i}s_1,\ r_2 + \mathrm{i}s_2,\ r_3 + \mathrm{i}s_3,\ r_4 + \mathrm{i}s_4), \\ &= |(r_t + \mathrm{i}s_t,\ \mathrm{i}r_x - s_x,\ \mathrm{i}r_y - s_y,\ \mathrm{i}r_z - s_z), \\ &= r_t + \mathrm{i}s_t,\ \mathrm{i}\mathbf{r} - \mathbf{s}, \end{aligned}\right\} \tag{2·18}$$

the last line being given in the usual notation for 3-vectors. Hence

$$q^* = |(r_t - \mathrm{i}s_t,\ -\mathrm{i}r_x - s_x,\ -\mathrm{i}r_y - s_y,\ -\mathrm{i}r_z - s_z). \tag{2·18a}$$

Using the matrix η of (2·1) we have $r^* = \eta r$, $s^* = \eta s$, and consequently $q^* = \eta(r - \mathrm{i}s)$, as against $q^* = \rho - \mathrm{i}\sigma$. ρ and σ are not vectors like x, however, their components all being real. By (2·18)

$$\rho = |(r_t, -s_x, -s_y, -s_z), \quad \sigma = |(s_t, r_x, r_y, r_z). \tag{2·19}$$

While the decomposition of a complex vector q into reals ρ and σ is useful in simplifying some calculations, its decomposition into simple-term vectors r and s is more useful when the object is to reduce results to standard tensor forms.

3. Field and mechanical representations of matter

(a) *Field-matrix representation of matter density*

In relativity theory the invariant *action* $\mathrm{d}A$ of the matter in a small volume

$$\mathrm{d}\tau = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,c\,\mathrm{d}t$$

of space-time is found, when all its factors are expressed as real positive numbers, to be

$$\mathrm{d}A = \rho_0 c \sqrt{-g}\,\mathrm{d}\tau,$$

where ρ_0 is the proper density of the matter, g the determinant of g_{ik}. Action is not, like matter, *in* space and time, but a single scalar concept that transcends both sides; in a deductive theory it is the most fundamental invariant conceivable, and we can imagine matter density ρ_0 and four-volume $\sqrt{-g}\,\mathrm{d}\tau$ as secondary invariants formed by decomposing it into factors. Hence, whatever view may be taken of the ultimate nature of the fundamental matter of the universe which classical theory assumes present, mostly formed into particles, it is legitimate to regard it here as a continuous basic substance of locally varying density existing in space and time.

The object here is to decompose a scalar function of position representing the density of this matter into two 4^2 matrices, of which in each case it is the resulting magnitude, one denoting a vector field similar to the electromagnetic field but made as general as possible, and the other the functions of this field which may be expected to express the mechanical properties of the matter. It will be useful to write $k\rho_0$, where k is a dimensional and possibly a numerical factor, for the 'density scalar' to be decomposed. (Another invariant that the resultant magnitude has been used in electromagnetic theory to define the mass density, and the assumption that k shall be taken to be 2 instead of unity is useful in discussing this.)

Let us first decompose $k\rho_0$ (expressed in energy units) into two scalar factors, as far as possible equal, consistent with generality. If we write

$$k\rho_0 = z_0 z_0^*, \tag{3·1}$$

the factors might be real and equal, imaginary and equal but of opposite signs, or full complex conjugates, having the dimensions (like electric or magnetic force) of (energy density)$^{\frac{1}{2}}$. By constructing now a 4^2 matrix in which the 11 term is $z_0 z_0^*$ and all the others zero, we obtain a form which may be written $z^0\overline{z^{0*}}$, since it is the matrix product of a column vector z^0 $(= |(z_0\,0\,0\,0)$ and a row vector $\overline{z^{0*}}$ $(= |(z_0^*\,0\,0\,0)$, each being parallel to the time axis 1, when the matrix is taken to represent a construction in Euclidean 4-space. Thus the fundamental matter at rest is represented by a vector field parallel to the time axis.

By applying a local rotation transformation to this matrix we should get a representation of the matter at the point it refers to in another state, as in the example of the mass vector of a particle. In doing this, however, two somewhat conflicting conditions must be satisfied at the same time.

(i) Since the general rotation Ω will convert z^0 and z^{0*}, even when they are simple originally, into complex vectors, we have to ensure that they remain conjugate, for it is only in this way that they can denote a single vector field; and (2) some characteristic scalar of the matrix must remain invariant and equal to $z_0 z_0^*$. The conditions require the resulting matrix to take the form

$$z\overline{z^*} = \Omega z^0\,\overline{\Omega^* z^{0*}} = \Omega z^0 \overline{z^{0*}}\Omega^*, \tag{3·2}$$

and the invariant to be the resultant magnitude of the matrix,

$$\{(z\overline{z^*})_{ik}\,(z\overline{z^*})_{ik}\}^{\frac{1}{2}} = (z_i z_i)^{\frac{1}{2}}\,(z_h^* z_k^*)^{\frac{1}{2}} = z_0 z_0^* = k\rho_0. \tag{3·3}$$

As may be seen from (2·16) the scalar of (3·3) forms the only way of expressing a *real* magnitude for a complex vector field subjected to orthogonal transformations; it also becomes clear why a square matrix $(z\overline{z^*})$ (and not simply a single vector z, since the magnitude of this may be complex) is required for the representation of matter by a vector field.

If now we divide the whole region of the fourfold representation into infinitesimal elements and rotate the vectors of each by (3·2), with the Ω an arbitrary function of x, they will mark out (when joined up again suitably) the curved lines of a vector field which represents the local density and state (whatever state it may be that the rotation denotes) of a distribution of matter which varies in space and time in as arbitrary a way as can be imagined. It will certainly not be matter as we know it until some further conditions are imposed; there is, for instance, no permanence assumed, and it is evident that restrictions of some sort will have to be imposed before this representation can be fruitful.

(b) *The classical energy tensor*

In relativity theory the mechanical properties of fluid matter in motion are expressed by a well-known 'energy' tensor

$$T^{ik} = \rho_0 c^2 \frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s} = \frac{\rho_0}{(1-v^2/c^2)}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}, \tag{3·4}$$

and a similar process to that employed above can be used to derive this. For this purpose, instead of factorizing $k\rho_0$, write the rest matrix of the matter density as $k\rho_0(1\,\overline{1^0})$, where 1^0

and $\overline{1^0}$ denote unit vectors parallel to x, in column and row forms. Further let the new state T be obtained by applying the same Lorentz rotation to each vector, so that

$$T = k\rho_0\,\bar{c}\mathrm{R}\,\bar{c}\mathrm{S}^{-1}(1^0\overline{1^0})\,\overline{\bar{c}\mathrm{R}\,\bar{c}\mathrm{S}^{-1}}.$$

Then, when the c_α are translated into the real quantities $v_x \ldots v_z$ by (2·12), this takes the form

$$T = \frac{k\rho_0}{1-\dfrac{v^2}{c^2}}\begin{bmatrix} 1 & (\mathrm{i})\,v_x/c & (\mathrm{i})\,v_y/c & (\mathrm{i})\,v_z/c \\ (\mathrm{i})\,v_x/c & (-)\,v_xv_x/c^2 & (-)\,v_xv_y/c^2 & (-)\,v_xv_z/c^2 \\ (\mathrm{i})\,v_y/c & (-)\,v_yv_x/c^2 & (-)\,v_yv_y/c^2 & (-)\,v_yv_z/c^2 \\ (\mathrm{i})\,v_z/c & (-)\,v_zv_x/c^2 & (-)\,v_zv_y/c^2 & (-)\,v_zv_z/c^2 \end{bmatrix}. \tag{3·5}$$

Equation (3·5) is identical with (3·4) when $k = c^2$ and (3·4) is expressed in terms of 'Euclidean' co-ordinates $x^i = (ct, \mathrm{i}x, \mathrm{i}y, \mathrm{i}z)$. The i's and minus signs enclosed in brackets indicate how the matrix of a tensor expressed on these co-ordinates differs from its contravariant value in terms of $x^i = (ct, x, y, z)$. There are advantages in the Euclidean form in this argument, the chief being that with it the contravariant and covariant values of the matrix are the same; hence, for example, its resultant magnitude may be obtained simply by taking the square root of the sum of the squares of all the terms. The physical significance of the fact that this is $k\rho_0$ is that the rest mass density of a moving fluid has been decomposed into 16 (although only 10 are different) component mechanical properties of the matter.

As regards the nature of the properties, wherever extended matter is in motion we can contemplate at any point, in addition to the energy density W and the momentum density **G**, two further quantities, the flux of energy, and the flux of momentum, at the point. In (3·5) the last two quantities are accounted for as the energy and momentum carried along by the matter, and measured by the rate at which these pass through a unit area fixed in space normal to the flow. In this way one is led to construct the energy-momentum-flux matrix **T** given below, the velocity of light c being introduced so as to make all the terms have the same dimensions as W

$$\mathbf{T} = \begin{bmatrix} W & Wv_x/c & Wv_y/c & Wv_z/c \\ cG_x & G_xv_x & G_xv_y & G_xv_z \\ cG_y & G_yv_x & G_yv_y & G_yv_z \\ cG_z & G_zv_x & G_zv_y & G_zv_z \end{bmatrix}. \tag{3·6}$$

Each row in (3·6) denotes the density of a quantity and the x-, y-, z-components of its vector flux (apart from c), the quantities being in successive rows energy and x-, y-, z-components of momentum. Since $\sqrt{-g}\,\mathrm{d}V c\,\mathrm{d}t$, where $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, is invariant, it is easily seen that $1/\sqrt{-g}\,\mathrm{d}V$ transforms as $c\,\mathrm{d}t$ or $\mathrm{d}x^1$, W $(= w/\sqrt{-g}\,\mathrm{d}V$, w = energy transforming like $x^1)$ as $\mathrm{d}x^1\mathrm{d}x^1$, G^i similarly as $\mathrm{d}x^i\mathrm{d}x^1$, v^k as $\mathrm{d}x^k/\mathrm{d}x^1$. Hence T^{ik} transforms like $\mathrm{d}x^i\mathrm{d}x^k$, and is a contravariant tensor. Its representation in the fourfold, where

$$x^i = (ct\,\mathrm{i}x\,\mathrm{i}y\,\mathrm{i}z) \quad (x = \zeta\mathrm{x})$$

consequently gives

$$T = \zeta\mathbf{T}\zeta. \tag{3·7}$$

This introduces the bracketed i's and signs of (3·5), and the result is identical with (3·5) if the energy density of the moving matter is taken to be $k\rho_0/(1-v^2/c^2)$.

Now just as in the previous case it is possible to suppose the Lorentz rotation producing (3·5) to be a function of x, and so imagine this matrix to represent the mechanical properties of a fluid moving locally in an arbitrary way. But before it can be expected to describe actual matter certain restrictions, essentially of conservation of energy and momentum, must be imposed on it; and these are found to require that in (3·5) the velocity v must everywhere be uniform.

(c) *The general 'mechanical properties' matrix*

By employing more velocities than v one can specify a matrix, Σ say), which describes arbitrary densities and fluxes of energy and momentum. Thus $v^{(1)} \ldots v^{(4)}$ being arbitrary velocities, in addition to the v of the moving matter which determines the momentum

$$\Sigma = \begin{bmatrix} W & (\mathrm{i})\, Wv_x^{(1)}/c & (\mathrm{i})\, Wv_y^{(1)}/c & (\mathrm{i})\, Wv_z^{(1)}/c \\ (\mathrm{i})\, cG_x & (-)\, G_x v_x^{(2)} & (-)\, G_x v_y^{(2)} & (-)\, G_x v_z^{(2)} \\ (\mathrm{i})\, cG_y & (-)\, G_y v_x^{(3)} & (-)\, G_y v_y^{(3)} & (-)\, G_y v_z^{(3)} \\ (\mathrm{i})\, cG_z & (-)\, G_z v_x^{(4)} & (-)\, G_z v_y^{(4)} & (-)\, G_z v_z^{(4)} \end{bmatrix}. \tag{3·8}$$

Σ is a representation in Euclidean co-ordinates, given by introducing the scheme of bracketed i's and signs (3·5), since in it momentum and velocity are antithetic to energy. The matrix, however, is no longer the outer product of a vector with itself; it is not a tensor but a matrix of 16 virtually arbitrary terms which does not keep its form after equal Lorentz rotations of its row and column vectors. It is clear that extensive limitations must be placed on the $v^{(\alpha)}$ before (3·8) can denote the properties of an element of fundamental matter, but assuming this done, we see that the matrix can be interpreted as rows showing energy and component momentum densities and their fluxes, as before, but now the fluxes must be pictured as traversing the matter, in so far as the $v^{(\alpha)}$ differ from v.

In classical physics matter in such a state is regarded as being under stress. Through any area imagined in the matter and moving along with it a force is being exerted producing, say, positive momentum on the far side of it. At the same time the reaction force produced an equal amount of negative momentum on the near side, which may cancel positive momentum originally there. In any case just the same effect is produced as if there were a bodily flow of positive momentum from the near to the far side through the area. Further the work done by the stress at the moving area produces effectively a flux of energy in a similar way.

Let P be a stress system carried along with the matter, then $\mathbf{P}_x$ (for example) denotes the *vector* thrust exerted forwards through a unit area perpendicular to x and moving with the matter, and P_{xx}, P_{xy}, P_{xz} are the components of $\mathbf{P}_x$. (Negative components denote tensions through the areas.)

When this stress is assumed present in the system (3·7) with v no longer necessarily uniform, the fluxes through an area fixed in space will be increased (of momentum by $\mathbf{P}_x$, and of energy by the scalar product $(\mathbf{vP}_x)$, $= v_x P_x + v_y P_{xy} + v_z P_{xz}$, through the unit area normal to x, etc.), and the matrix T will be changed to one of the general type Σ, in which

$$\Sigma = \begin{bmatrix} W & \{(\mathrm{i})/c\}\{Wv_x + (\mathbf{vP}_x)\} & \{(\mathrm{i})/c\}\{Wv_y + (\mathbf{vP}_y)\} & \{(\mathrm{i})/c\}\{Wv_z + (\mathbf{vP}_z)\} \\ (\mathrm{i})\, cG_x & (-)\,(G_x v_x + P_{xx}) & (-)\,(G_x v_y + P_{yx}) & (-)\,(G_x v_z + P_{zx}) \\ (\mathrm{i})\, cG_y & (-)\,(G_y v_x + P_{xy}) & (-)\,(G_y v_y + P_{yy}) & (-)\,(G_y v_z + P_{zy}) \\ (\mathrm{i})\, cG_z & (-)\,(G_z v_x + P_{xz}) & (-)\,(G_z v_y + P_{yz}) & (-)\,(G_z v_z + P_{zz}) \end{bmatrix}. \tag{3·9}$$

Since $c\mathbf{G} = (\mathbf{v}/c)\, W$, the 16 arbitrary variables of (3·8) have been reduced to 13 in (3·9), but there is still ample scope to impose the further restrictions required.

Let us finally write the single symbol $\mathbf{S}$ (components $\mathrm{S}_x, \mathrm{S}_y, \mathrm{S}_z$) for the vector flux of energy in (3·9), and denote the momentum flux components in terms of a single stress system Π, i.e. $\Pi_{xy} = G_y v_x + P_{xy}$, etc.; then

$$\Sigma = \begin{bmatrix} W & (\mathrm{i})\,\mathrm{S}_x/c & (\mathrm{i})\,\mathrm{S}_y/c & (\mathrm{i})\,\mathrm{S}_z/c \\ (\mathrm{i})\,cG_x & (-)\,\Pi_{xx} & (-)\,\Pi_{yx} & (-)\,\Pi_{zx} \\ (\mathrm{i})\,cG_y & (-)\,\Pi_{xy} & (-)\,\Pi_{yy} & (-)\,\Pi_{zy} \\ (\mathrm{i})\,cG_z & (-)\,\Pi_{xz} & (-)\,\Pi_{yz} & (-)\,\Pi_{zz} \end{bmatrix}. \tag{3·10}$$

In (3·10) the 3^2 matrix Π (without the minus signs), measuring the fluxes of momentum through areas fixed in space, is a stress system of the stationary type envisaged by Maxwell in the electromagnetic field. Being stationary, it can do no work, and so cannot account for the energy flux; this has therefore to be entered in the matrix as a separate concept, $\mathbf{S}$, as the Poynting flux is in electromagnetic theory.

Σ is evidently a very wide form capable of describing the most varied mechanical states of fundamental matter imaginable, whether these are possible or impossible in Nature. It is also important that the velocity of bodily motion that was present at the start has disappeared from open expression; we are, therefore, no longer obliged to think of the changes that take place as being produced by the movement of identifiable matter, but can regard them as caused by the rise and fall with time in related ways of the 16 component terms of Σ. This, of course, is essential for a field theory of matter.

(*d*) *Relation between field and mechanical matrices*

The matrix (3·10) is well known in electromagnetic theory, but I hope that the method of approach to it made here will not be considered unduly long, because it has been required in order to bring out some features of the matrix not ordinarily recognized. The three forms of Σ, (3·8), (3·9) and (3·10) emphasize different aspects of the matrix; it denotes in the first a set of four rows of densities and corresponding fluxes, in the second a material fluid moving under stress, and in the third a set of densities and stresses varying at each point with the time, as in the theory of the electromagnetic field.

It is evident that there is no difference in principle between the Σ of (3·8) and the T of (3·5); consequently, in laying down the restriction that the (otherwise arbitrary) Σ should apply to fundamental matter, the first requisite is that the resultant magnitude of Σ shall be required to remain the same as its value, $k\rho_0$, in T. This fits in well with the classical idea that a stress system is devoid of mass, for this condition by itself requires that the stress system P of (3·9) must be limited in such a way that when superposed on T it does not alter the proper density of the matter. The same argument shows, however, that the Maxwellian stress Π in (3·10) is *not* devoid of mass; it requires to be included in the calculation of the mass density, $k\rho_0$, and this is a feature of (3·10) which is not usually taken into account.

When the condition that the resultant magnitude $k\rho_0$ is to be kept unchanged is maintained, the change from T to Σ can be looked on as a very general kind of transformation, for it converts the representation of the same portion of matter from one state T in which the fluxes are solely due to convection, to another Σ in which they are of the most general kind possible. In view of this it is perhaps not surprising to find that the rest mass density scalar $k\rho_0$ *can* be decomposed into a matrix of the Σ-type by a transformation which corresponds to that by which the field matrix $(\overline{zz^*})$ of (3·2) was developed; in consequence there exists also a general transformation which converts one of these matrices into the other.

This transformation, which has been studied in a previous paper (Milner 1952), where it was called an '*e*-transformation', is as follows: for any 4^2 matrix A an '*e*-transform' (denoted by ${}^eA$) may be obtained by evaluating either of the expressions (readily proved to be equivalent by (2·6))

$$ {}^eA = \tfrac{1}{2}A_{\alpha\beta}\mathrm{R}_\alpha\mathrm{S}_\beta = \tfrac{1}{2}\mathrm{S}_j A\mathrm{R}_j. \tag{3·11}$$

The detailed value given below,

$$ {}^eA = \tfrac{1}{2}\begin{bmatrix} (A_{11}+A_{22})+(A_{33}+A_{44}) & (A_{12}-A_{21})-(A_{34}-A_{43}) & (A_{13}-A_{31})+(A_{24}-A_{42}) & (A_{14}-A_{41})-(A_{23}-A_{32}) \\ -(A_{12}-A_{21})-(A_{34}-A_{43}) & (A_{11}+A_{22})-(A_{33}+A_{44}) & (A_{23}+A_{32})-(A_{14}+A_{41}) & (A_{24}+A_{42})+(A_{13}+A_{31}) \\ -(A_{13}-A_{31})+(A_{24}-A_{42}) & (A_{23}+A_{32})+(A_{14}+A_{41}) & (A_{11}+A_{33})-(A_{22}+A_{44}) & (A_{34}+A_{43})-(A_{12}+A_{21}) \\ -(A_{14}-A_{41})-(A_{23}-A_{32}) & (A_{24}+A_{42})-(A_{13}+A_{31}) & (A_{34}+A_{43})+(A_{12}+A_{21}) & (A_{11}+A_{44})-(A_{22}+A_{33}) \end{bmatrix} \tag{3·12}$$

shows that ${}^eA$ is also a certain type of *transpose* of A. After expressing each term of A as the sum of its symmetric and antisymmetric parts, $\tfrac{1}{2}(A_{ik}+A_{ki})$ and $\tfrac{1}{2}(A_{ik}-A_{ki})$.. $(i<k)$, interchange $(A_{12}+A_{21})$ with $-(A_{34}-A_{43})$, $(A_{13}+A_{31})$ with $+(A_{24}-A_{42})$, $(A_{14}+A_{41})$ with $-(A_{23}-A_{32})$, and the non-diagonal terms of ${}^eA$ are obtained. The diagonal terms also are formed on a simple principle. The resultant magnitude is unaffected by the transformation (${}^eA_{ik}{}^eA_{ik} = A_{ik}A_{ik}$); also a repetition of the process brings us back to A again; thus A *and* ${}^eA$ *are e-transforms of each other*.

In the case of the field matrix $(\overline{zz^*})$ the *e*-transformation gives

$$ {}^e(\overline{zz^*}) = \tfrac{1}{2}z_\alpha z_\beta \mathrm{R}_\beta\mathrm{S}_\beta = \tfrac{1}{2}\bar{z}\mathrm{R}\,\overline{z^*}\mathrm{S} = Z, \quad \text{say.} \tag{3·13}$$

This produces a remarkable change in the character of the matrix, and in the physical interpretation it admits. The non-diagonal terms of $(\overline{zz^*})$ are complex, but in Z the imaginary parts of these have become shifted to the first row and first column, excluding the 11 term, in such a way as to make these wholly imaginary, and all other terms wholly real. The rows now denote four simple-term vectors, of which the first is antithetic to the other three, so that they can stand for energy and momentum densities and fluxes. Z in fact takes on exactly the form required for the Σ of (3·8). The row vectors are not arbitrary, however; it follows from (3·13) and (2·7) that they are equal in magnitude and mutually perpendicular. This limitation reduces the independent variables of Z to the same number as those of $(\overline{zz^*})$.

These considerations serve to justify the basic assumption of the analysis. This is that the rest-mass density scalar $k\rho_0$ of fundamental matter can be decomposed into, or alternatively regarded as the resultant of, the component terms of two matrices, $(\overline{zz^*})$ and Z, the first denoting in 4-space a complex vector field, and the second its energy-momentum-flux mechanical properties.

The assumption is confirmed by evaluating Z in detail, when a correspondence between it and the similar matrix of standard electromagnetic theory becomes apparent. To bring this out, write as in (2·18)

$$\left.\begin{aligned} z = e+\mathrm{i}h &= |(e_t+\mathrm{i}h_t,\quad \mathrm{i}e_x-h_x,\ \ldots,\quad \mathrm{i}e_z-h_z),\\ z^* = e^*-\mathrm{i}h^* &= |(e_t-\mathrm{i}h_t,\ -\mathrm{i}e_x-h_x,\ \ldots,\ -\mathrm{i}e_z-h_z),\end{aligned}\right\} \tag{3·14}$$

when $(\overline{zz^*})$ can readily be evaluated, and then Z $(= {}^e(\overline{zz^*}))$ from (3·12). The Z so obtained is given below, but for convenience it is separated into two parts, ${}^tZ$ and ${}^rZ$, ${}^tZ$ consisting of all the parts of terms which contain the suffix t, and ${}^rZ$ of the remainder.

$$Z = {}^tZ + {}^rZ, \tag{3·15}$$

where

$${}^tZ = \begin{bmatrix} \frac{1}{2}(e_t^2+h_t^2) & -\mathrm{i}(e_te_x+h_th_x) & -\mathrm{i}(e_te_y+h_th_y) & -\mathrm{i}(e_te_z+h_th_z) \\ \mathrm{i}(e_te_x+h_th_x) & \frac{1}{2}(e_t^2+h_t^2) & e_th_z-e_zh_t & -(e_th_y-e_yh_t) \\ \mathrm{i}(e_te_y+h_th_y) & -(e_th_z-e_zh_t) & \frac{1}{2}(e_t^2+h_r^2) & e_th_x-e_xh_t \\ \mathrm{i}(e_te_z+h_th_z) & e_th_y-e_yh_t & -(e_th_x-e_xh_t) & \frac{1}{2}(e_t^2+h_t^2) \end{bmatrix},$$

$${}^rZ = \begin{bmatrix} \frac{1}{2}(e_x^2+e_y^2+e_z^2+h_x^2+h_y^2+h_z^2) & \mathrm{i}(e_yh_z-e_zh_y) & \mathrm{i}(e_zh_x-e_xh_z) & \mathrm{i}(e_xh_y-e_yh_x) \\ \mathrm{i}(e_yh_z-e_zh_y) & \frac{1}{2}(e_x^2-e_y^2-e_z^2+h_x^2-h_y^2-h_z^2) & e_xe_y+h_xh_y & e_xe_z+h_xh_z \\ \mathrm{i}(e_zh_x-e_xh_z) & e_xe_y+h_xh_y & \frac{1}{2}(-e_x^2+e_y^2-e_z^2-h_x^2+h_y^2-h_z^2) & e_ye_z+h_yh_z \\ \mathrm{i}(e_xh_y-e_yh_x) & e_xe_z+h_xh_z & e_ye_z+h_yh_z & \frac{1}{2}(-e_x^2-e_y^2+e_z^2-h_x^2-h_y^2+h_z^2 \end{bmatrix}.$$

Examination of (3·15) shows that ${}^rZ$ is identical with the standard value of the mechanical matrix Σ of an electromagnetic field $e_x\,..\,h_z$ when these are reckoned in Heaviside units, and note is taken of the fact that Z includes the i's and signs of (3·10) due to its being a representation in Euclidean 4-space.

$e+\mathrm{i}h$ is, however, a 4-vector field representing a completely arbitrary distribution of matter, and it contains components e_t and h_t which are not included in the standard electromagnetic field. These give rise to the part ${}^tZ$ of Z, which has some significant features complementary to those of ${}^rZ$. Thus, part from the diagonal terms, ${}^tZ$ constitutes the antisymmetric part of Z. Again each term ${}^tZ_{ik}$ has a different pair of suffixes t, x, y, z from the corresponding ${}^rZ_{ik}$; it thus represents another kind of component associated with the dual plane, which provides an unsuspected place for it, so to speak, in the theoretical picture. As a result of these properties ${}^tZ$ and ${}^rZ$ satisfy a general condition of perpendicularity

$${}^tZ_{ik}{}^rZ_{ik} = 0, \tag{3·16a}$$

while the squares of their respective magnitudes simply add together to form that of Z

$${}^tZ_{ik}{}^tZ_{ik} + {}^rZ_{ik}{}^rZ_{ik} = Z_{ik}Z_{ik}. \tag{3·16b}$$

All this forms evidence that ${}^{t}Z$ is soundly formulated as the extension of the standard ${}^{r}Z$ to take e_t and h_t into account, and so far as I can see it is the only systematic one possible. But it must be observed that the formula (3·13) defining Z is not necessarily unique, for other forms might have been chosen which, while retaining the resultant magnitude invariant, give results different in detail (e.g. ${}^{e}(z^*\bar{z})$, $\frac{1}{2}\mathrm{R}_j(z\overline{z^*})\,\mathrm{S}_j$, etc.). An explanation is that electromagnetic theory contains many conventional definitions, such as the signs of **e** and **h** in relation to charge, axes, etc., and an examination of a number of these forms shows that they are merely variants of this sort, which can be brought to the form (3·15) by suitable identification of symbols.

4. Relation of field equations of matter to electromagnetic theory

(a) *The field equations*

So far the field of e and h considered has been taken to be arbitrary in the 4-space; its mechanical properties Z_{ik} vary therefore arbitrarily with time, as well as through space. In classical theory, where we are not yet concerned with uncertainty, an arbitrary time variation, though imaginable, must be taken as impossible in Nature. A system passes from one configuration to another by a definite 'natural' path, which is determined by applying the principle of stationary action. The principle might be applied either to the e, h field directly, or to the equivalent picture of fluid matter in motion under stress. The first method is more laborious because electromagnetic tubes in four dimensions are required for discussing the variations, but the two methods have been shown to lead to identical conclusions in the case of the electromagnetic field where free of charge (Milner 1928). It is consequently simpler to rely on the second method here, since this can now be used by our basic assumption of the complete equivalence of the e, h field with the matter. The principle leads at once to the classical laws of motion, which state that the rates of increase of the momentum, and of the energy, or any element of fluid matter are given by the resultant force on it, and the activity, of the accompanying P-type moving stress of (3·9). A further direct development is the conclusion that, if the fluid matter is isolated from external forces, the laws of motion may be equivalently expressed by the equations

$$\frac{\partial W}{\partial t} = -\frac{\partial \mathrm{S}_k}{\partial x_k}, \quad \frac{\partial G_i}{\partial t} = -\frac{\partial \Pi_{ki}}{\partial x_k} \quad \begin{pmatrix} i, k = x, y, z \\ x_k = x, y, z \end{pmatrix}. \tag{4·1}$$

These are well-known equations in electromagnetic theory—the first states that the rate of increase of the energy in a unit volume fixed in space is equal to the net rate at which energy is carried out of the volume by the flux **S**; the second that the rate of increase of each component of positive momentum in the unit volume is equal to that component of the resultant force of the stress Π on the volume.

In what follows partial differentiation with respect to x_i will be denoted by ∂_i, and a column and a row of the four ∂_i by ∂ and $\tilde{\partial}$, respectively. When converted into space-time terms

$$\partial = |(\partial_1 \,..\, \partial_4) = \left|\left(\frac{\partial}{\mathrm{i}\,\partial t}, \frac{\partial}{\mathrm{i}\,\partial x}, \frac{\partial}{\mathrm{i}\,\partial y}, \frac{\partial}{\mathrm{i}\,\partial z}\right)\right., \tag{4·2a}$$

which will be abbreviated to

$$\partial = |(\partial_{it}, -\mathrm{i}\partial_x, -\mathrm{i}\partial_y, -\mathrm{i}\partial_z), \tag{4·2b}$$

while its conjugate
$$\partial^* = |(\partial_{it}, +\mathrm{i}\partial_x, +\mathrm{i}\partial_y, +\mathrm{i}\partial_z) = \eta\partial. \tag{4·2c}$$

When equations (4·1) are converted into their representations in the fourfold, by writing

$$\left.\begin{array}{l} W = \Sigma_{11}, \quad G_i = \dfrac{1}{ic}\Sigma_{ir}, \quad \mathrm{S}_k = \dfrac{c}{\mathrm{i}}\Sigma_{1k}, \quad \Pi_{ki} = -\Sigma_{ik}, \quad \dfrac{\partial}{\partial t} = c\,\partial_1, \quad \dfrac{\partial}{\partial x_k} = \mathrm{i}\,\partial_k \\ \qquad (\mathrm{i}, k = x, y, z \text{ in } G, \mathrm{S}, \Pi = 2, 3, 4 \text{ in } \Sigma, \partial). \end{array}\right\} \tag{4·3}$$

in accordance with (3·10) and (4·2), they become expressed by the single form

$$\partial_k \Sigma_{ik} = 0 \quad (i, k = 1\,..\,4). \tag{4·4}$$

Since Z is a special case of the general Σ, the equation

$$\partial_k Z_{ik} = 0 \tag{4·5}$$

denotes the *limitation which the principle of action places on the otherwise arbitrary values imaginable of the variation with time of the mechanical quantities contained in* Z, and therefore on the variables $e_t\,..\,e_z$, $h_t\,..\,h_z$, of the field of basic matter.

Physically, (4·5) means that conservation of energy and of each component of momentum holds in the field; geometrically the meaning is that each of the 4-vectors formed by the rows of Z has zero divergence—a property that enables it to be visualized in the form of a tube of constant flux in 4-space.

To apply the condition (4·5) we can write, by (3·13) and (2·6*d*), Z in the two forms

$$Z = \tfrac{1}{2}(\bar{z}\mathrm{R})\,(\overline{z^*\mathrm{S}}) = \tfrac{1}{2}(\overline{z^*\mathrm{S}})\,(\bar{z}\mathrm{R}),$$

and use the second and first of these expressions for differentiating the z and z^* factors, respectively. This gives

$$\partial_k Z_{ik} = \tfrac{1}{2}(\overline{z^*\mathrm{S}})_{ij}\,\partial_k(\bar{z}\mathrm{R})_{jk} + \tfrac{1}{2}(\bar{z}\mathrm{R})_{ij}\,\partial_k(\overline{z^*\mathrm{S}})_{jk} = 0. \tag{4·6}$$

Now, by (2·6*b*) and (4·2*c*),

$$\partial_k(\overline{z^*\mathrm{S}})_{jk} = \partial_k(z^*_\beta \eta_{jl} \mathrm{R}_{\beta lm} \eta_{mk}) = \partial^*_m(\eta_{jl} z^*_\beta \mathrm{R}_{\beta lm});$$

consequently (4·6) becomes

$$\tfrac{1}{2}(\overline{z^*\mathrm{S}})_{ij}\,\partial_k(\bar{z}\mathrm{R})_{jk} + \tfrac{1}{2}(\bar{z}\mathrm{R})_{ij}\,\eta_{jl}\partial^*_m(\overline{z^*\mathrm{R}})_{lm} = 0.$$

This is a set of four (one for each value of i) non-linear differential equations which an otherwise arbitrary complex vector field in 4-space must satisfy in order that it may account fully for the mechanical behaviour of fundamental matter required by classical relativity mechanics. If we write

$$\partial_k(\bar{z}\mathrm{R})_{jk} = -q_j, \quad \partial^*_m(\overline{z^*\mathrm{R}})_{lm} = -q^*_l, \tag{4·7}$$

then either of these conjugate equations defines a derived vector field q, (q^*), which is completely determined when the field z, (z^*) is given; and these two fields must satisfy the condition

$$-\tfrac{1}{2}\{(\overline{z^*\mathrm{S}})_{ij}\,q_j + (\bar{z}\mathrm{R})_{ij}\,(\eta q^*)_j\} = 0. \tag{4·8}$$

In this way (4·6) has been split into two sets of equations, one of which is differential (and now linear), and lays no restrictions whatever on z, but merely defines a field q derived from z, and the other is a set of restrictions in the form of four algebraical relations between z, z^* and q, q^*. (The suffixes in these equations are all written below, in the way usual with matrix formulae. It will be proved later that (4·7) and (4·8) are tensor equations.) These equations may now be dealt with separately.

(b) *The extended electromagnetic equations*

Taking (4·7) first, putting $q = r+\mathrm{i}s$, and writing the first equation

$$\partial_k\{(\bar{e}+\mathrm{i}\bar{h})\,\mathrm{R}\}_{ik}+(r+\mathrm{i}s)_i = 0, \tag{4·9}$$

we can convert this into its space-time form by using formulae (4·2), (2·7), (3·14), (2·18). It then becomes the set of four complex equations given below

$$\left.\begin{aligned}(\partial_{ct}e_t-\partial_x e_x-\partial_y e_y-\partial_z e_z+r_t)+\mathrm{i}(\partial_{ct}h_t-\partial_x h_x-\partial_y h_y-\partial_z h_z+s_t) &= 0,\\ \mathrm{i}(\partial_{ct}e_x-\partial_x e_t-\partial_y h_z+\partial_z h_y+r_x)-(\partial_{ct}h_x-\partial_x h_t+\partial_y e_z-\partial_z e_y+s_x) &= 0,\\ \mathrm{i}(\partial_{ct}e_y-\partial_y e_t-\partial_z h_x+\partial_x h_z+r_y)-(\partial_{ct}h_y-\partial_y h_t+\partial_z e_x-\partial_x e_z+s_y) &= 0,\\ \mathrm{i}(\partial_{ct}e_z-\partial_z e_t-\partial_x h_y+\partial_y h_x+r_z)-(\partial_{ct}h_z-\partial_z h_t+\partial_x e_y-\partial_y e_x+s_z) &= 0.\end{aligned}\right\} \tag{4·10}$$

Equation (4·10) is identical with the standard electromagnetic equations, apart from the extra terms in e_t, h_t, $s_t \ldots s_z$; it is evident, however, that extra terms are required to describe an *arbitrary* vector field in 4-space. The relation to the standard equations becomes still more marked on denoting by single symbols the e_t and r, and the h_t and s terms. Let

$$\left.\begin{aligned}j_t &= r_t+\partial_{ct}e_t, & k_t &= s_t+\partial_{ct}h_t,\\ j_x &= r_x-\partial_x e_t, & k_x &= s_x-\partial_x h_t,\\ j_y &= r_y-\partial_y e_t, & k_y &= s_y-\partial_y h_t,\\ j_z &= r_z-\partial_z e_t, & k_z &= s_z-\partial_z h_t.\end{aligned}\right\} \tag{4·10a}$$

Then, on substituting these in it, equating separately to zero the real and imaginary parts, and expressing it in the usual 3-vector notation, (4·10) becomes

$$\left.\begin{aligned}\operatorname{div}\mathbf{e} = j_t &= r_t+\frac{1}{c}\frac{\partial e_t}{\partial t},\\ \frac{1}{c}\frac{\partial \mathbf{e}}{\partial t}-\operatorname{curl}\mathbf{h} &= -\mathbf{j} = -\{\mathbf{r}-\operatorname{grad}e_t\},\\ \operatorname{div}\mathbf{h} = k_t &= s_t+\frac{1}{c}\frac{\partial h_t}{\partial t},\\ \frac{1}{c}\frac{\partial \mathbf{h}}{\partial t}+\operatorname{curl}\mathbf{e} &= -\mathbf{k} = -\{\mathbf{s}-\operatorname{grad}h_t\}.\end{aligned}\right\} \tag{4·11}$$

The two left-hand parts are now exactly the Maxwell–Lorentz equations with j_t, $\mathbf{j}$ denoting electric charge and current densities, except for the inclusion of magnetic charge and current densities, k_t, $\mathbf{k}$. *The essence of the extensions contained in* (4·10) *has now been concentrated into the* right-hand parts of (4·11), which define charge and current densities in a more general way than does standard theory.

In the classical theory the six variables $e_x \ldots h_z$ are derived from a 4-potential Φ and the same applies to the eight variables $e_t \ldots h_z$ of the extended equations as a result of the following simple proposition:

Let $\phi = \psi+\mathrm{i}\chi$ be an arbitrary complex vector field, and q a field defined by

$$q = \bar{\partial}\partial\phi, \tag{4·12a}$$

then, if z is another field, defined by $\quad z = \bar{\partial}\mathrm{R}\phi, \qquad (4·12b)$

since $\bar{\partial}R^{-1}\partial R = \bar{\partial}\partial 1$ we shall have the equation

$$\bar{\partial}R^{-1}z - q = 0. \tag{4·12c}$$

The last equation when expressed in full is (4·10), and the defining equations of e, h, r and s are given by (4·12a, b) as

$$\left.\begin{aligned} e_t + \mathrm{i}h_t &= (\partial_{ct}\psi_t - \partial_x\psi_x - \partial_y\psi_y - \partial_z\psi_z) + \mathrm{i}(\partial_{ct}\chi_t - \partial_x\chi_x - \partial_y\chi_y - \partial_z\chi_z),\\ \mathrm{i}e_x - h_x &= \mathrm{i}(\partial_{ct}\psi_x - \partial_x\psi_t + \partial_y\chi_z - \partial_z\chi_y) - (\partial_{ct}\chi_x - \partial_x\chi_t - \partial_y\psi_z + \partial_z\psi_y),\\ \mathrm{i}e_y - h_y &= \mathrm{i}(\partial_{ct}\psi_y - \partial_y\psi_t + \partial_z\chi_x - \partial_x\chi_z) - (\partial_{ct}\chi_y - \partial_y\chi_t - \partial_z\psi_x + \partial_x\psi_z),\\ \mathrm{i}e_z - h_z &= \mathrm{i}(\partial_{ct}\psi_z - \partial_z\psi_t + \partial_z\chi_y - \partial_y\chi_x) - (\partial_{ct}\chi_z - \partial_z\chi_t - \partial_x\psi_y + \partial_y\psi_x),\\ (r_t \,..\, r_z, s_t \,..\, s_z) &= (\partial_{ct}^2 - \partial_x^2 - \partial_y^2 - \partial_z^2)(\psi_t \,..\, \psi_z, \chi_t \,..\, \chi_z). \end{aligned}\right\} \tag{4·12d}$$

The corresponding formulae for $e_x \,..\, h_z$ in the classical theory are

$$e_x = -\partial_{ct}\Phi_x - \partial_x\Phi_t, \quad h_x = \partial_y\Phi_z - \partial_z\Phi_y,$$

etc., in cyclic order. In the classical theory Φ has 4 real space-time components. Equation (4·12d) is evidently its extension to a complex 4-potential. This fact affects the equations defining e, h and r, but not j, which is defined in accordance with the classic form.

(c) *The conservation restrictions on the equations*

In so far as they are not limited unwittingly by the background of physical ideas which we associate with the names 'electric and magnetic force, charge and current', equations (4·11) hold identically in an entirely arbitrary complex vector field in 4-space. This field, however, must be subjected to the conservation restrictions (4·8) before it can serve as a satisfactory representation of fundamental matter. (4·8) may be reduced to a somewhat simpler form as follows:

If we write it as
$$z^*_\beta S_{\beta ij}\, q_j + z_\alpha (R_\alpha \eta)_{il} + q^*_l = 0,$$
it can be converted by (2·6c) into
$$q_j R_{ij\beta} z^*_\beta + z_\alpha R_{i\alpha l} q^*_l = 0,$$
which can be written in the shorter matrix form
$$\bar{q}(R_i z^*) + \bar{z}(R_i q^*) = 0. \tag{4·13a}$$

Each of these terms denotes a scalar product of two vectors, e.g. of q and the permuted and sign-changed vector $R_i z^*$ derived from z^*. Converted to space and time terms the equations become

$$\left.\begin{aligned} e_t r_t + (\mathbf{er}) + h_t S_t + (\mathbf{hS}) &= 0 \quad \text{for} \quad i = 1 \qquad (a)\\ e_t\mathbf{r} + \mathbf{e}r_t + [\mathbf{rh}] + h_t\mathbf{s} + \mathbf{h}S_t - [\mathbf{se}] &= 0 \quad \text{for} \quad (i = 2 \,..\, 4) \qquad (b) \end{aligned}\right\} \tag{4·13b}$$

where $(..)$ denotes the scalar, and $[..]$ the vector product of the included space vectors, $\mathbf{e} = (e_x, e_y, e_z)$, etc.

It is interesting at this stage to see how these restrictions operate in electromagnetic theory. Let us consider separately the question of the conservation of the energy in the two parts ${}^tZ$ and ${}^rZ$ of the energy-momentum-flux matrix (3·12). Taking ${}^rZ$ first, it is easy to verify from (4·10), (4·11), that

$$\begin{aligned} c\,\partial_k\,{}^rZ_{1k} &= \frac{\partial}{\partial t}\tfrac{1}{2}(e_x^2 + e_y^2 + e_z^2 + h_x^2 + h_y^2 + h_z^2) + \frac{\partial}{\partial x}c(e_y h_z - e_z h_y) + \frac{\partial}{\partial y}c(e_z h_x - e_x h_z)\\ &\quad + \frac{\partial}{\partial z}c(e_x h_y - e_y h_x) = -c(e_x j_x + e_y j_y + e_z j_z + h_x k_x + h_y k_y + h_z k_z). \end{aligned} \tag{4·14}$$

This is a well-known classical formula when $k = 0$, and it is sometimes referred to as if it indicated conservation of energy in the field, whereas except in fields free of currents it states just the opposite of this. It states that the rate at which the energy in a fixed unit volume is increasing with the time (the first term of the middle part), *plus* the rate at which energy is flowing out of the volume (the next three terms), is not zero, but equal to the term on the right. This term, when it is, as here, negative, can only indicate a rate of destruction of electromagnetic energy inside the volume. We owe to Lorentz its interpretation as the rate at which the field loses energy by doing work on the current, assumed now to be constituted by moving electrons. The energy lost by the field appears again in the form of extra kinetic energy of the electrons, acted on by the Lorentz mechanical force, but, since the current carried by the electrons is accompanied by a magnetic field, one must assume that some at any rate of the energy lost is ultimately radiated back into the field. Now this is a valuable method of describing what goes on where it is a matter of visible charged bodies moving in an electric field, when the Lorentz force is in fact deducible from observation, but it is not a method which can logically be applied in a formative theory of the electron. If one has to assume the presence in the electron of mechanical energy not wholly accounted for by the field, the attempt to formulate a field theory of matter has failed. On the other hand, if *all* the energy of the electron is to be regarded as electromagnetic in character, the classical formula (4·22) becomes hopelessly inadequate. The electrons in the unit volume must in this case be included in the electromagnetic system; their energies and fluxes must be accounted for completely by the left-hand side of the equation, and the only value of the right-hand side consistent with conservation of energy is zero. It is clear that this condition cannot be satisfied unless we admit the existence of energy and flux additional to those recognized by the standard theory.

Turning now to the ${}^tZ$ of (3·15), it may readily be verified from (4·10), (4·11), as before, but now also making direct use of the conservation condition equation (4·13*a*), that

$$c\,\partial_k{}^tZ_{ik} = \frac{\partial}{\partial t}\tfrac{1}{2}(e_t^2+h_t^2)+\frac{\partial}{\partial x}c(e_te_x+h_th_x)+\frac{\partial}{\partial y}c(e_te_y+h_th_y)+\frac{\partial}{\partial z}c(e_te_z+h_th_z)$$
$$= +c(e_xj_x+e_yj_y+e_zj_z+h_xk_x+h_yk_y+h_zk_z). \qquad (4{\cdot}15)$$

Let us distinguish the quantities here and those in (4·14) by the prefixes t and r, respectively. Then (4·15) denotes that the rate at which t-energy is increasing in the fixed unit volume (first term), *plus* the rate at which t-energy is flowing out of the volume (the next three terms), is not zero, and the positive expression on the right indicates a rate of creation of t-energy in the volume. This is identical with the rate at which r-energy is being destroyed in the volume, so that, when the energies and fluxes of ${}^tZ$ are included along with those of ${}^rZ$ in the field system, the total energy is rigorously conserved.

Similar results are obtained for the momentum components from the three other rows of Z. Combined they show that the excess of the rate of increase of the momentum in a unit volume over the resultant force on the volume of the Maxwellian stress is respectively *minus* and *plus* the Lorentz force $(\mathbf{e}j_t+[\mathbf{jh}]+\mathbf{h}k_t-[\mathbf{ke}])$ for r- and t-momentum and stress—the latter result again requiring the use of (4·13). Thus here also the Lorentz force now operates conservatively in the field, by transferring momentum from the r- to the t-form (and conversely when the force is negative); and the transference occurs only in the places where charge and current are situated.

There is of course nothing very surprising in these conservation results; we are bound to get them in some form for conservation has been imposed on the equations as a condition. Nevertheless, there is considerable interest in the way in which the condition works out. The condition not only indicates the necessity of extensions in the electromagnetic equations but *itself, when applied to* Z, *formulates the extended scheme*. Secondly, conservation holds for classical *r*-energy and momentum and for *t*-energy and momentum separately at all places in the field which are free of charge or current. In these places the classical equations are completely satisfactory, it is only where charge and current are present that they show signs of breaking down. *It is also at these latter places alone that interchanges between r- and t-forms of energy and momentum come into play, explaining and at the same time remedying the weakness of the classical account.* Finally, the rates of interchange are expressed by the activity and action of a mechanical force as originally formulated by Lorentz. If electrons are regarded as small charged particles of 'matter' set into motion by the action of the Lorentz force of the field, *the justification for this common practical treatment will not be affected in any way*. It is only in fundamental theory, or if one seeks to account for the electron's mass as an electromagnetic property, that the additional terms in the energy-stress matrix appear to require open recognition.

(*d*) *Summary*

This paper has been concerned with the deduction from first principles, as briefly as possible subject to making the meaning clear, of the field equations which would be expected to be characteristic of fundamental matter. These are found to consist of the set of pairs of equations (4·11) (by themselves applying to a completely arbitrary field), which are subjected to the restrictions on the values of the variables (4·13) required to express conservation. The left-hand set of the equations (4·11) are also found to be identical with the electromagnetic equations in their standard form, provided they are subject to a restriction that

$$k_t = 0, \quad \mathbf{k} = 0, \tag{4·16}$$

i.e. that magnetic charge and current are zero. (The two sets of restrictions (4·13) and (4·16) are consistent, and may be assumed in operation together.) In the standard electromagnetic equations, however, the right-hand side set of equations of (4·11) does not appear to have been considered, yet it brings out the possibility of the existence of the extra variables, e_t and h_t, one at least of which is essential if the conservation restrictions are to be obeyed.

Some matters which have been left over are the tensor character of the full equations, the four potential and the mass density of the extended electromagnetic field, and the enablement which the extension gives of formulating a purely electromagnetic theory of the mass of the electron. It is hoped to deal with these in a later communication.

References

Milner, S. R. 1928 *Proc. Roy. Soc.* A, **120**, 483.
Milner, S. R. 1952 *Proc. Roy. Soc.* A, **214**, 292, 312 (referred to as A and B).
Pauli, W. 1921 *Relativitätstheorie*. Leipzig : B. G. Teubner.
Turnbull, H. W. 1929 *Theory of determinants, matrices and invariants*, p. 157. London: Blackie and Son.